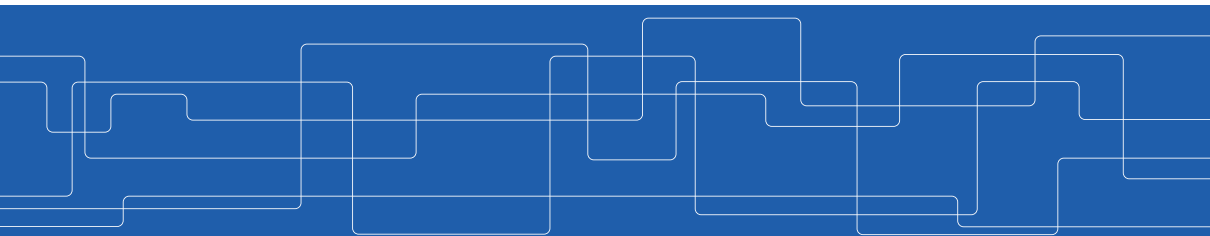




# Autoencoders and Restricted Boltzmann Machines

Amir H. Payberah  
payberah@kth.se  
2020-12-01



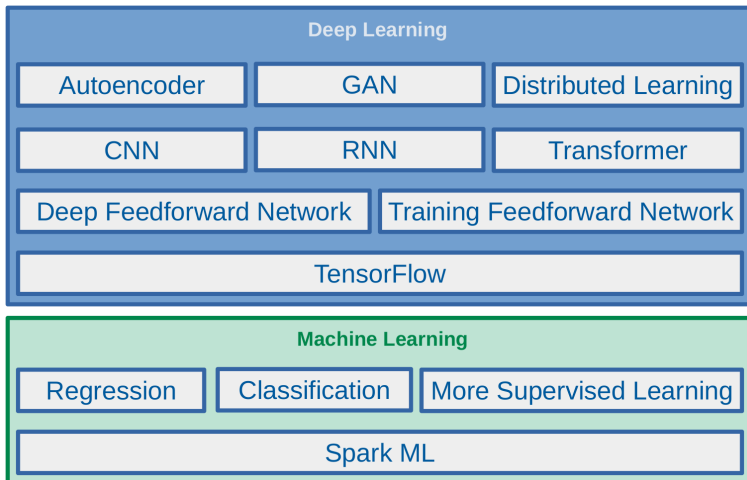


## The Course Web Page

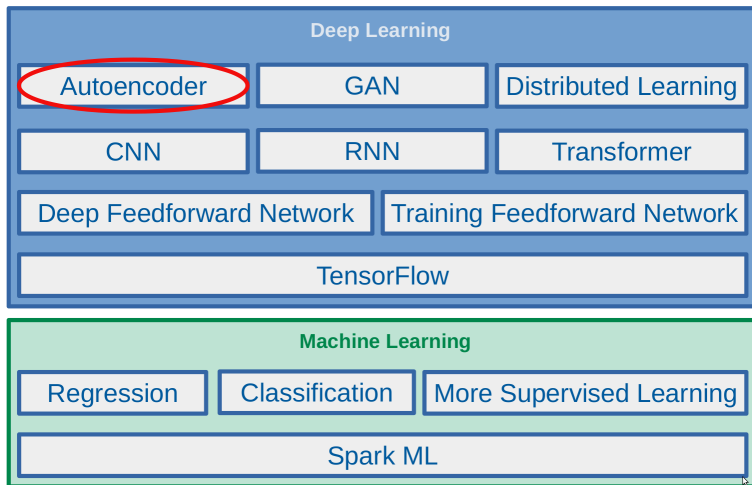
`https://id2223kth.github.io`  
`https://tinyurl.com/y6kcpmzy`



# Where Are We?



# Where Are We?





# Let's Start With An Example



- ▶ Which of them is **easier to memorize**?



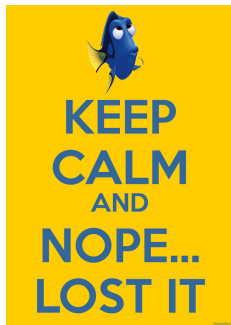
- ▶ Which of them is **easier to memorize?**
- ▶ **Seq1:** 40, 27, 25, 36, 81, 57, 10, 73, 19, 68

- ▶ Which of them is **easier to memorize**?
- ▶ **Seq1**: 40, 27, 25, 36, 81, 57, 10, 73, 19, 68
- ▶ **Seq2**: 50, 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20



Seq1 : 40, 27, 25, 36, 81, 57, 10, 73, 19, 68

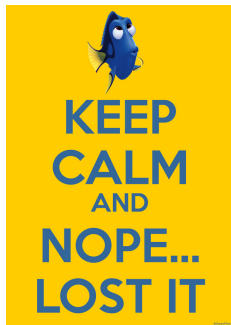
Seq2 : 50, 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20



Seq1 : 40, 27, 25, 36, 81, 57, 10, 73, 19, 68

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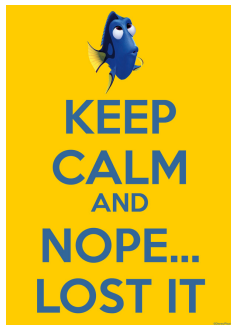
- ▶ Seq1 is shorter, so it should be easier.



Seq1 : 40, 27, 25, 36, 81, 57, 10, 73, 19, 68

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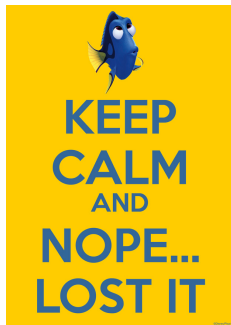
- ▶ Seq1 is shorter, so it should be easier.
- ▶ But, Seq2 follows two simple rules:



Seq1 : 40, 27, 25, 36, 81, 57, 10, 73, 19, 68

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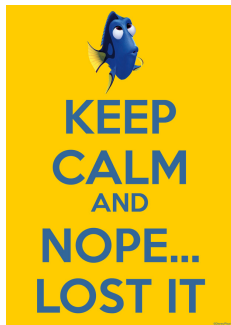
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  - Even numbers are followed by their half.
  - Odd numbers are followed by their triple plus one.



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- ▶ But, Seq2 follows two simple rules:
  - Even numbers are followed by their half.
  - Odd numbers are followed by their triple plus one.
- ▶ You don't need pattern if you could quickly and easily memorize very long sequences
- ▶ But, it is hard to memorize long sequences that makes it useful to recognize patterns.



- ▶ 1970, W. Chase and H. Simon
- ▶ They observed that **expert chess players** were able to **memorize** the positions of **all the pieces in a game** by looking at the board for just **5 seconds**.



- ▶ This was only the case when the pieces were placed in realistic positions, not when the pieces were placed randomly.





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- ▶ Chess experts don't have a much better memory than you and I.



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- ▶ This was only the case when the pieces were placed in realistic positions, not when the pieces were placed randomly.
- ▶ Chess experts don't have a much better memory than you and I.
- ▶ They just see chess patterns more easily due to their experience with the game.
- ▶ Patterns helps them store information efficiently.

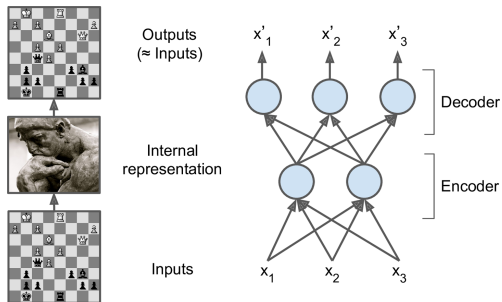




# Autoencoders

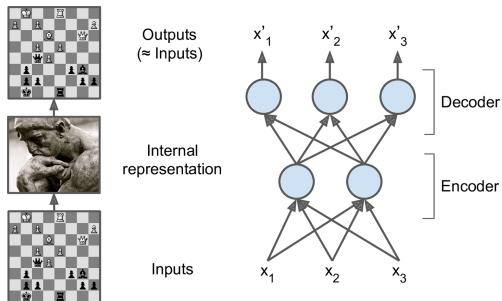
# Autoencoders (1/5)

- ▶ Just like the chess players in this memory experiment.



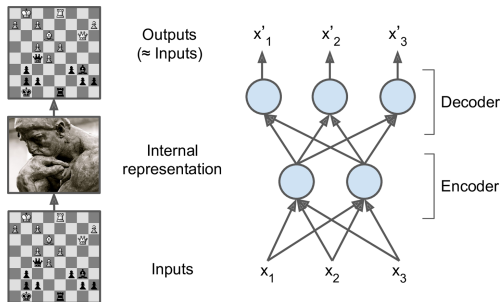
# Autoencoders (1/5)

- ▶ Just like the chess players in this memory experiment.
- ▶ An **autoencoder** looks at the inputs, **converts** them to an **efficient internal representation**, and then **spits out** something that **looks very close to the inputs**.



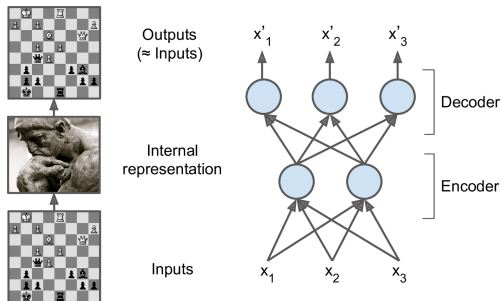
# Autoencoders (2/5)

- The same **architecture** as a **Multi-Layer Perceptron (MLP)**.



# Autoencoders (2/5)

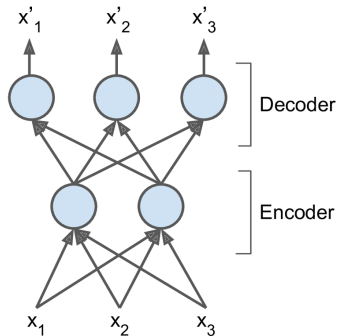
- ▶ The same **architecture** as a **Multi-Layer Perceptron (MLP)**.
- ▶ Except that the number of **neurons in the output layer** must be **equal** to the **number of inputs**.





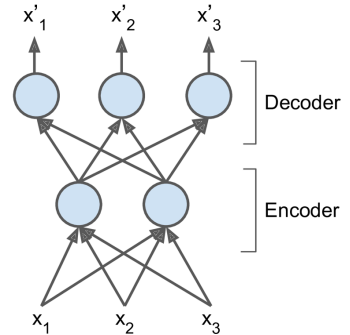
## Autoencoders (3/5)

- ▶ An **autoencoder** is always composed of **two parts**.



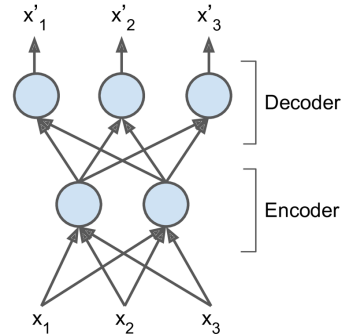
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- ▶ An **encoder (recognition network)**,  $\mathbf{h} = f(\mathbf{x})$   
Converts the **inputs to an internal representation**.



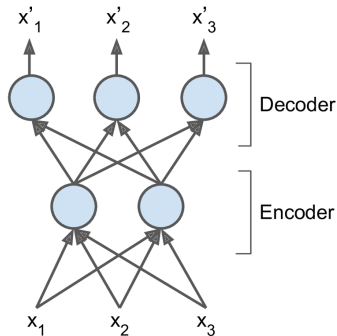
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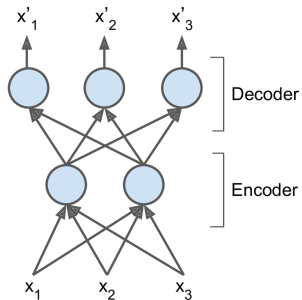
# Autoencoders (3/5)

- ▶ An **autoencoder** is always composed of **two parts**.
- ▶ An **encoder (recognition network)**,  $\mathbf{h} = \mathbf{f}(\mathbf{x})$   
Converts the **inputs** to an **internal representation**.
- ▶ A **decoder (generative network)**,  $\mathbf{r} = \mathbf{g}(\mathbf{h})$   
Converts the **internal representation** to the **outputs**.
- ▶ If an autoencoder learns to set  $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$  everywhere, it is **not especially useful, why?**



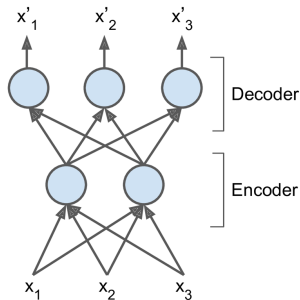
# Autoencoders (4/5)

- ▶ Autoencoders are designed to be **unable to learn to copy perfectly**.



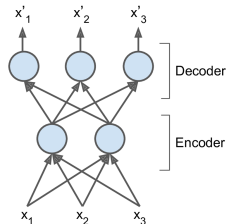
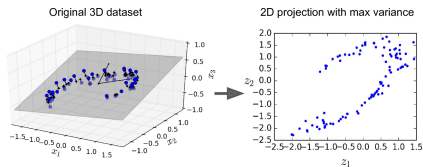
# Autoencoders (4/5)

- ▶ Autoencoders are designed to be **unable to learn to copy perfectly**.
- ▶ The models are forced to **prioritize which aspects of the input should be copied**, they often learn **useful properties** of the data.



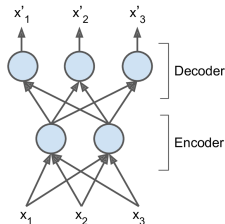
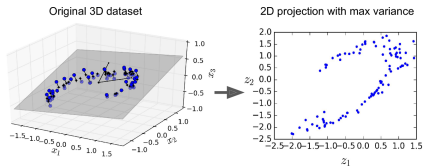
# Autoencoders (5/5)

- ▶ **Autoencoders** are neural networks capable of learning **efficient representations** of the **input data** (called **codings**) **without any supervision**.



# Autoencoders (5/5)

- ▶ **Autoencoders** are neural networks capable of learning **efficient representations** of the **input data** (called **codings**) **without any supervision**.
- ▶ **Dimension reduction**: these **codings** typically have a much **lower dimensionality** than the **input data**.







## Different Types of Autoencoders

- ▶ Stacked autoencoders
- ▶ Denoising autoencoders
- ▶ Sparse autoencoders
- ▶ Variational autoencoders

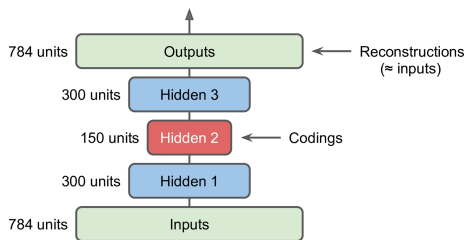


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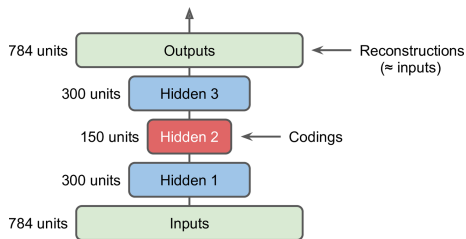
# Stacked Autoencoders (1/3)

- ▶ **Stacked autoencoder:** autoencoders with **multiple hidden layers**.



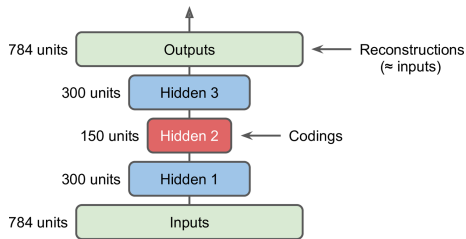
## Stacked Autoencoders (1/3)

- ▶ **Stacked autoencoder**: autoencoders with **multiple hidden layers**.
- ▶ Adding **more layers** helps the autoencoder learn more **complex codings**.



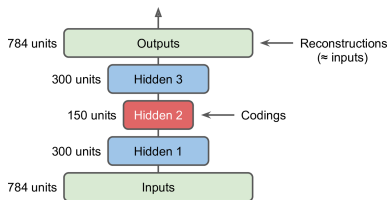
# Stacked Autoencoders (1/3)

- ▶ **Stacked autoencoder**: autoencoders with **multiple hidden layers**.
- ▶ Adding **more layers** helps the autoencoder learn more **complex codings**.
- ▶ The architecture is typically **symmetrical** with regards to the **central hidden layer**.



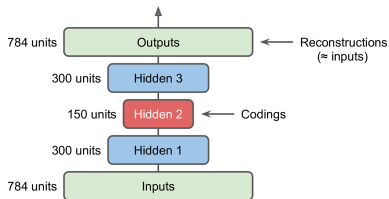
## Stacked Autoencoders (2/3)

- ▶ In a symmetric architecture, we can **tie the weights** of the **decoder** layers to the weights of the **encoder** layers.



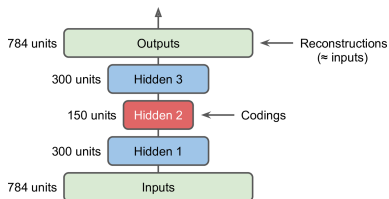
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- ▶ In a symmetric architecture, we can **tie the weights** of the **decoder** layers to the weights of the **encoder** layers.
- ▶ In a network with  $N$  layers, the **decoder layer weights** can be defined as  $w_{N-1+1} = w_1^T$ , with  $1 = 1, 2, \dots, \frac{N}{2}$ .



## Stacked Autoencoders (2/3)

- ▶ In a symmetric architecture, we can **tie the weights** of the **decoder** layers to the weights of the **encoder** layers.
- ▶ In a network with  $N$  layers, the **decoder layer weights** can be defined as  $w_{N-1+1} = w_1^T$ , with  $1 = 1, 2, \dots, \frac{N}{2}$ .
- ▶ This **halves** the **number of weights** in the model, **speeding up training** and **limiting the risk of overfitting**.







## Stacked Autoencoders (3/3)

```
stacked_encoder = keras.models.Sequential([
    keras.layers.Flatten(input_shape=[28, 28]),
    keras.layers.Dense(100, activation="relu"),
    keras.layers.Dense(30, activation="relu"),
])
stacked_decoder = keras.models.Sequential([
    keras.layers.Dense(100, activation="relu", input_shape=[30]),
    keras.layers.Dense(28 * 28, activation="sigmoid"),
    keras.layers.Reshape([28, 28])
])

model = keras.models.Sequential([stacked_encoder, stacked_decoder])
```

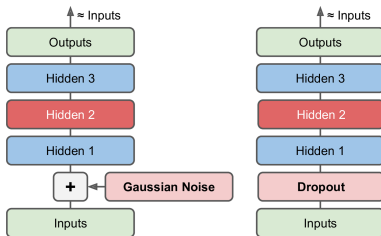


## Different Types of Autoencoders

- ▶ Stacked autoencoders
- ▶ **Denoising autoencoders**
- ▶ Sparse autoencoders
- ▶ Variational autoencoders

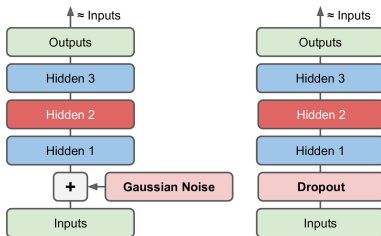
# Denoising Autoencoders (1/4)

- ▶ One way to force the autoencoder to learn useful features is to add noise to its inputs, training it to recover the original noise-free inputs.



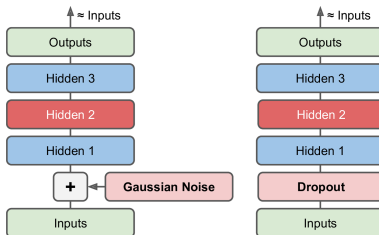
## Denoising Autoencoders (1/4)

- ▶ One way to force the autoencoder to **learn useful features** is to **add noise** to its **inputs**, training it to **recover the original noise-free inputs**.
- ▶ This prevents the autoencoder from **trivially copying its inputs to its outputs**, so it ends up having to find patterns in the data.



## Denoising Autoencoders (2/4)

- ▶ The noise can be pure **Gaussian noise** added to the inputs, or it can be **randomly switched off inputs**, just like in **dropout**.





## Denoising Autoencoders (3/4)

```
denoising_encoder = keras.models.Sequential([
    keras.layers.Flatten(input_shape=[28, 28]),
    keras.layers.Dropout(0.5),
    keras.layers.Dense(100, activation="relu"),
    keras.layers.Dense(30, activation="relu")
])
denoising_decoder = keras.models.Sequential([
    keras.layers.Dense(100, activation="relu", input_shape=[30]),
    keras.layers.Dense(28 * 28, activation="sigmoid"),
    keras.layers.Reshape([28, 28])
])
model = keras.models.Sequential([denoising_encoder, denoising_decoder])
```



## Denoising Autoencoders (4/4)

```
denoising_encoder = keras.models.Sequential([
    keras.layers.Flatten(input_shape=[28, 28]),
    keras.layers.GaussianNoise(0.2),
    keras.layers.Dense(100, activation="relu"),
    keras.layers.Dense(30, activation="relu")
])
denoising_decoder = keras.models.Sequential([
    keras.layers.Dense(100, activation="relu", input_shape=[30]),
    keras.layers.Dense(28 * 28, activation="sigmoid"),
    keras.layers.Reshape([28, 28])
])
model = keras.models.Sequential([denoising_encoder, denoising_decoder])
```



## Different Types of Autoencoders

- ▶ Stacked autoencoders
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## Sparse Autoencoders (1/2)

- ▶ Adding an appropriate term to the **cost function** to push the autoencoder to **reducing** the number of **active neurons** in the **coding layer**.



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- ▶ This forces the autoencoder to represent each input as a combination of a **small number of activations**.



## Sparse Autoencoders (1/2)

- ▶ Adding an appropriate term to the **cost function** to push the autoencoder to **reducing** the number of **active neurons** in the **coding layer**.
- ▶ This forces the autoencoder to represent each input as a combination of a **small number of activations**.
- ▶ As a result, **each neuron** in the **coding layer** typically ends up representing a **useful feature**.



## Sparse Autoencoders (2/2)

```
sparse_l1_encoder = keras.models.Sequential([
    keras.layers.Flatten(input_shape=[28, 28]),
    keras.layers.Dense(100, activation="selu"),
    keras.layers.Dense(300, activation="sigmoid", activity_regularizer=keras.regularizers.l1(1e-3))
])

sparse_l1_decoder = keras.models.Sequential([
    keras.layers.Dense(100, activation="selu", input_shape=[300]),
    keras.layers.Dense(28 * 28, activation="sigmoid"),
    keras.layers.Reshape([28, 28])
])

model = keras.models.Sequential([sparse_l1_encoder, sparse_l1_decoder])
```



## Different Types of Autoencoders

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- ▶ **Variational autoencoders**



## Variational Autoencoders (1/6)

- ▶ Variational autoencoders are probabilistic autoencoders.



## Variational Autoencoders (1/6)

- ▶ Variational autoencoders are probabilistic autoencoders.
- ▶ Their outputs are partly determined by chance, even after training.
  - As opposed to denoising autoencoders, which use randomness only during training.



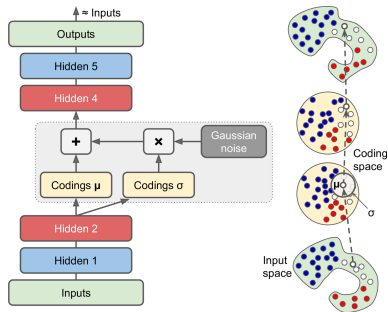
## Variational Autoencoders (1/6)

- ▶ **Variational autoencoders** are **probabilistic autoencoders**.
- ▶ Their outputs are **partly determined by chance**, **even after training**.
  - As opposed to denoising autoencoders, which use **randomness only during training**.
- ▶ They are **generative autoencoders**, meaning that they can **generate new instances** that look like they were sampled from the training set.



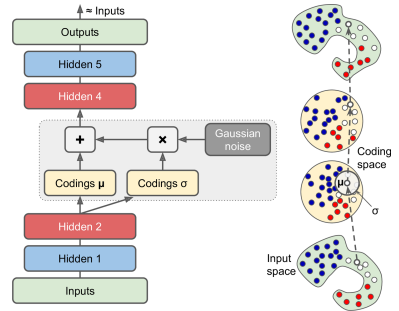
## Variational Autoencoders (2/6)

- ▶ Instead of directly producing a coding for a given input, the **encoder** produces a **mean coding  $\mu$**  and a **standard deviation  $\sigma$** .



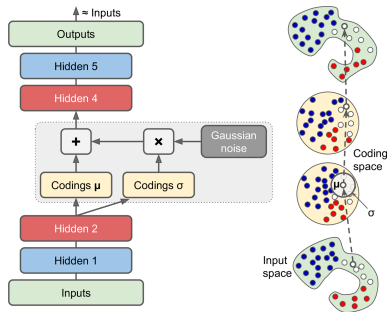
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- ▶ Instead of directly producing a coding for a given input, the **encoder** produces a **mean coding  $\mu$**  and a **standard deviation  $\sigma$** .
- ▶ The **actual coding** is then **sampled randomly** from a **Gaussian distribution** with **mean  $\mu$**  and **standard deviation  $\sigma$** .
- ▶ After that the **decoder** just **decodes the sampled coding normally**.





## Variational Autoencoders (3/6)

- ▶ The **cost function** is composed of **two parts**.



## Variational Autoencoders (3/6)

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- ▶ 1. the usual **reconstruction loss**.
  - Pushes the autoencoder to **reproduce its inputs**.
  - Using **cross-entropy**.



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  - Pushes the autoencoder to have **codings** that look as though they were **sampled from a simple Gaussian distribution**.



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  - Using the **KL divergence** between the **target distribution** (the Gaussian distribution) and the **actual distribution** of the codings.



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  - Using **cross-entropy**.
- ▶ 2. the **latent loss**
  - Pushes the autoencoder to have **codings** that look as though they were **sampled from a simple Gaussian distribution**.
  - Using the **KL divergence** between the **target distribution** (the Gaussian distribution) and the **actual distribution** of the codings.
  - $latent\_loss = -\frac{1}{2} \sum_1^K (1 + \log(\sigma_i^2) - \sigma_i^2 - \mu_i^2)$





## Variational Autoencoders (4/6)

### ► Encoder part

```
inputs = keras.layers.Input(shape=[28, 28])
z = keras.layers.Flatten()(inputs)
z = keras.layers.Dense(150, activation="relu")(z)
z = keras.layers.Dense(100, activation="relu")(z)
codings_mean = keras.layers.Dense(10)(z)
codings_log_var = keras.layers.Dense(10)(z)
codings = Sampling()([codings_mean, codings_log_var]) # normal distribution
variational_encoder = keras.models.Model(inputs=[inputs], outputs=[codings])
```



## Variational Autoencoders (5/6)

### ► Decoder part

```
decoder_inputs = keras.layers.Input(shape=[codings_size])
x = keras.layers.Dense(100, activation="relu")(decoder_inputs)
x = keras.layers.Dense(150, activation="relu")(x)
x = keras.layers.Dense(28 * 28, activation="sigmoid")(x)
outputs = keras.layers.Reshape([28, 28])(x)
variational_decoder = keras.models.Model(inputs=[decoder_inputs], outputs=[outputs])
```



## Variational Autoencoders (6/6)

```
codings = variational_encoder(inputs)
reconstructions = variational_decoder(codings)
model = keras.models.Model(inputs=[inputs], outputs=[reconstructions])

latent_loss = -0.5 * K.sum(1 + codings_log_var - K.exp(codings_log_var)
                          - K.square(codings_mean), axis=-1)
model.add_loss(K.mean(latent_loss) / 784.)
```

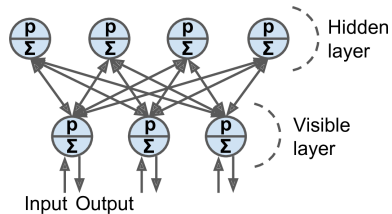




# Restricted Boltzmann Machines

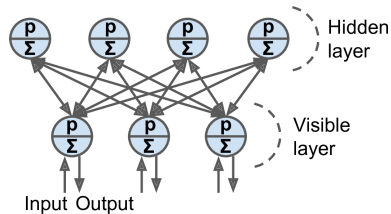
# Restricted Boltzmann Machines

- ▶ A Restricted Boltzmann Machine (RBM) is a stochastic neural network.



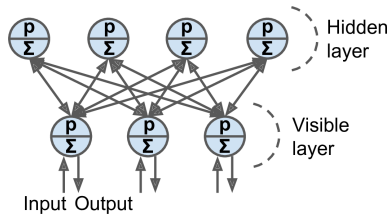
# Restricted Boltzmann Machines

- ▶ A **Restricted Boltzmann Machine (RBM)** is a **stochastic neural network**.
- ▶ **Stochastic** meaning these **activations** have a **probabilistic element**, instead of deterministic functions, e.g., logistic or ReLU.



# Restricted Boltzmann Machines

- ▶ A **Restricted Boltzmann Machine (RBM)** is a **stochastic neural network**.
- ▶ **Stochastic** meaning these **activations** have a **probabilistic element**, instead of deterministic functions, e.g., logistic or ReLU.
- ▶ The neurons form a **bipartite graph**:
  - One **visible** layer and one **hidden** layer.
  - A **symmetric connection** between the two layers.
  - There are **no connections** between neurons **within** a layer.







# Let's Start With An Example

# RBM Example (1/11)

- ▶ We have a set of **six movies**, and we ask users to tell us which ones **they want to watch**.



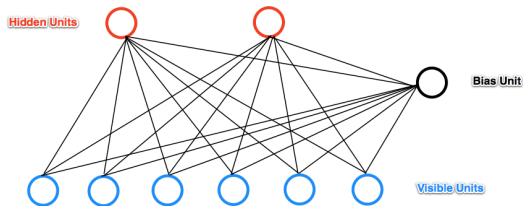
## RBM Example (1/11)

- ▶ We have a set of **six movies**, and we ask users to tell us which ones **they want to watch**.
- ▶ We want to learn two **latent neurons (hidden neurons)** underlying movie preferences, e.g., **SF/fantasy** and **Oscar winners**



# RBM Example (2/11)

- ▶ Our RBM would look like the following.



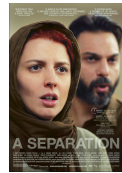
## RBM Example (3/11)

- ▶ Alice: (HP=1, Avatar=1, LOTR=1, Glad=0, Titan=0, Sep=0), Big SF fan.



## RBM Example (3/11)

- ▶ Alice: (HP=1, Avatar=1, LOTR=1, Glad=0, Titan=0, Sep=0), Big SF fan.
- ▶ Bob: (HP=1, Avatar=0, LOTR=1, Glad=0, Titan=0, Sep=0), SF fan, but not Avatar.



## RBM Example (3/11)

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## RBM Example (3/11)

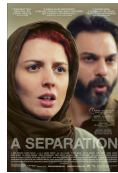
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## RBM Example (3/11)

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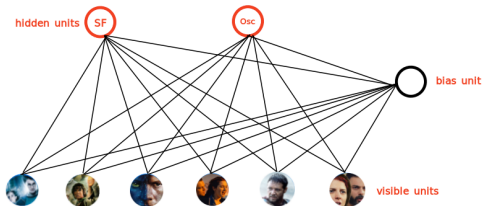
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## RBM Example (4/11)

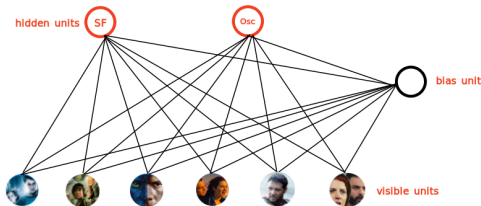
- ▶ Assume the given input  $x_i$  is the 0 or 1 for each visible neuron  $v_i$ .
  - 1: like a movie, and 0: dislike a movie



# RBM Example (4/11)

- ▶ Assume the given input  $x_i$  is the 0 or 1 for each visible neuron  $v_i$ .
  - 1: like a movie, and 0: dislike a movie
- ▶ Compute the activation energy at hidden neuron  $h_j$ :

$$a(h_j) = \sum_i w_{ij} v_i$$

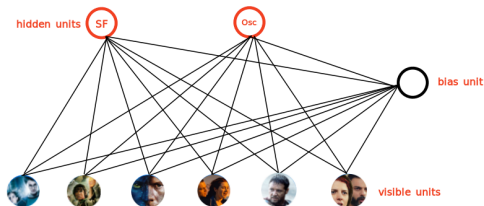


# RBM Example (5/11)

- ▶ For each hidden neuron  $h_j$ , we compute the **probability**  $p(h_j)$ .

$$a(h_j) = \sum_i w_{ij} v_i$$

$$p(h_j) = \text{sigmoid}(a(h_j)) = \frac{1}{1 + e^{-a(h_j)}}$$



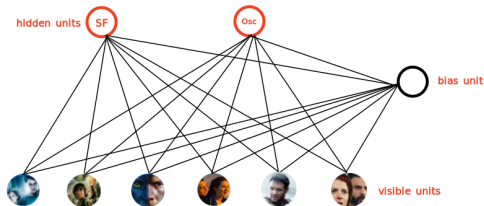
# RBM Example (5/11)

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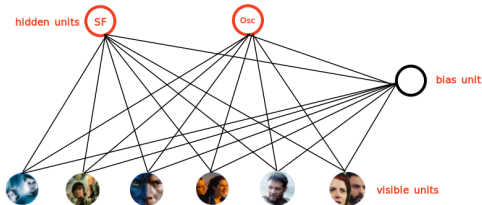
$$p(h_j) = \text{sigmoid}(a(h_j)) = \frac{1}{1 + e^{-a(h_j)}}$$

- ▶ We **turn on** the hidden neuron  $h_j$  with the **probability**  $p(h_j)$ , and **turn it off** with probability  $1 - p(h_j)$ .



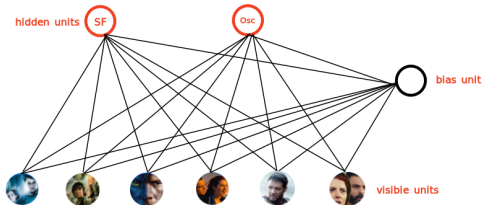
## RBM Example (6/11)

- ▶ Declaring that you like **Harry Potter**, **Avatar**, and **LOTR**, doesn't guarantee that the **SF/fantasy** hidden neuron will **turn on**.



## RBM Example (6/11)

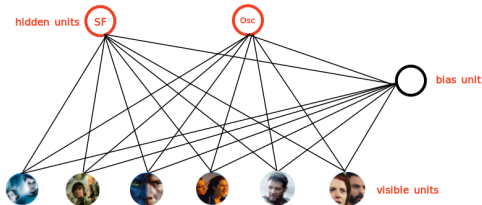
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- ▶ But it **will turn on** with a **high probability**.





## RBM Example (6/11)

- ▶ Declaring that you like **Harry Potter**, **Avatar**, and **LOTR**, doesn't guarantee that the **SF/fantasy** hidden neuron will **turn on**.
- ▶ But it **will turn on** with a **high probability**.
  - In reality, if you want to watch all three of those movies makes us highly suspect you like **SF/fantasy** in general.
  - But there's a **small chance** you like them for other reasons.





## RBM Example (7/11)

- ▶ Conversely, if we know that one person likes SF/fantasy (so that the SF/fantasy neuron is on)



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- ▶ The **hidden neurons** send messages to the **visible (movie) neurons**, telling them to **update their states**.

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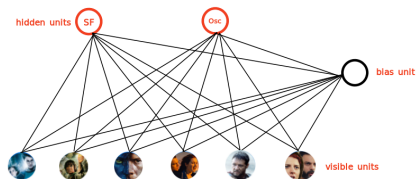
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- ▶ Being on the **SF/fantasy** neuron **doesn't guarantee** that we'll always recommend all three of **Harry Potter, Avatar, and LOTR**.
  - For example **not everyone** who likes science fiction liked Avatar.

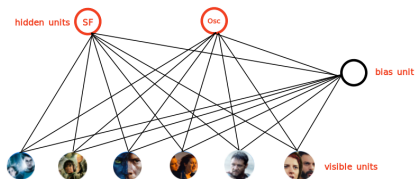
# RBM Example (8/11)

- ▶ How do we **learn** the **connection weights**  $w_{ij}$  in our network?



## RBM Example (8/11)

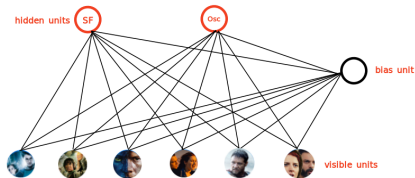
- ▶ How do we **learn** the **connection weights**  $w_{ij}$  in our network?
- ▶ Assume, as an input we have a bunch of **binary vectors**  $x$  with **six elements** corresponding to a **user's movie preferences**.





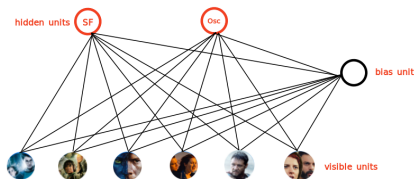
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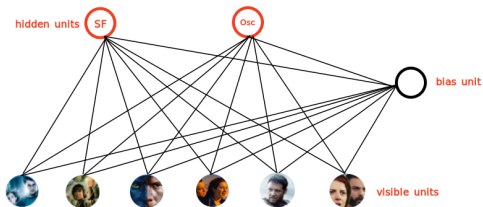
## RBM Example (8/11)

- ▶ How do we **learn** the **connection weights**  $w_{ij}$  in our network?
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- ▶ We do the **following steps** in each **epoch**:
- ▶ 1. Take a **training instance**  $x$  and set the **states** of the **visible neurons** to these preferences.



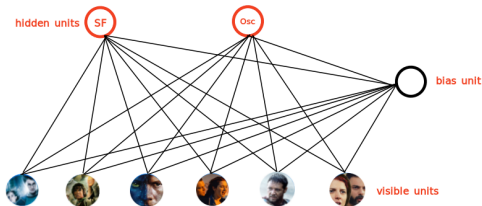
## RBM Example (9/11)

- ▶ 2. Update the **states** of the **hidden neurons**.



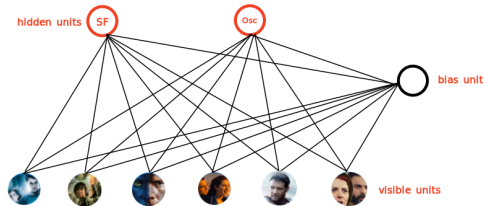
# RBM Example (9/11)

- ▶ 2. Update the **states** of the **hidden neurons**.
  - Compute  $a(\mathbf{h}_j) = \sum_i w_{ij} v_i$  for each **hidden neuron**  $\mathbf{h}_j$ .



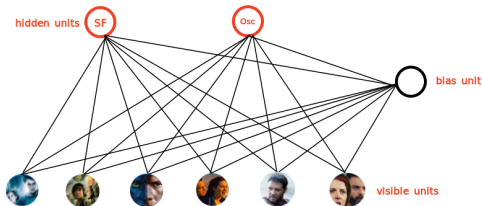
# RBM Example (9/11)

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- ▶ 3. For each edge  $e_{ij}$ , compute  $\text{positive}(e_{ij}) = v_i \times h_j$ 
  - I.e., for each **pair of neurons**, measure whether they are **both on**.



## RBM Example (10/11)

- ▶ 4. Update the **state** of the **visible neurons** in a similar manner.



# RBM Example (10/11)

- ▶ 4. Update the **state** of the **visible neurons** in a similar manner.
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- ▶ 5. Update the **hidden neurons** again similar to step 2. We denote the **updated hidden neurons** with  $h'_j$ .



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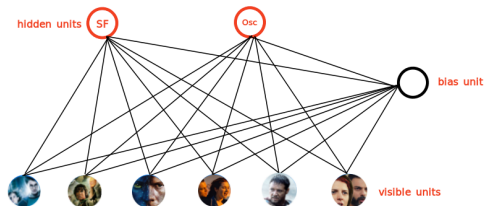
- ▶ 4. Update the **state** of the **visible neurons** in a similar manner.
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- ▶ 5. Update the **hidden neurons** again similar to step 2. We denote the **updated hidden neurons** with  $h'_j$ .
- ▶ 6. For each edge  $e_{ij}$ , compute  $\text{negative}(e_{ij}) = v'_i \times h'_j$



# RBM Example (11/11)

- ▶ 7. Update the weight of each edge  $e_{ij}$ .

$$w_{ij} = w_{ij} + \eta(\text{positive}(e_{ij}) - \text{negative}(e_{ij}))$$

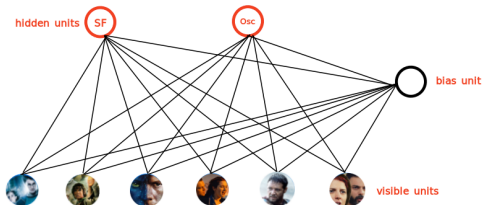


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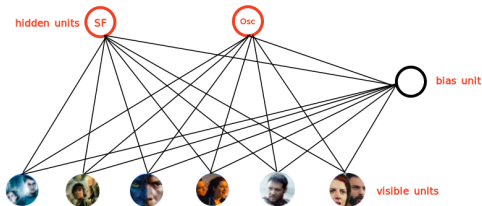


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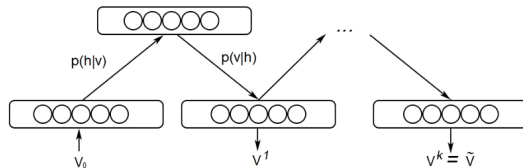
$$w_{ij} = w_{ij} + \eta(\text{positive}(e_{ij}) - \text{negative}(e_{ij}))$$

- ▶ 8. Repeat over all training examples.
- ▶ 9. Continue until the error between the training examples and their reconstructions falls below some threshold or we reach some maximum number of epochs.



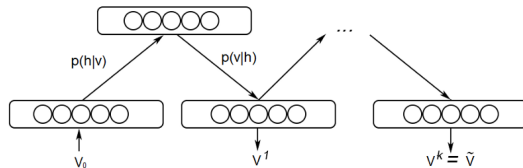
# RBM Training (1/2)

- ▶ Step 1, Gibbs sampling: what we have done in steps 1-6.



# RBM Training (1/2)

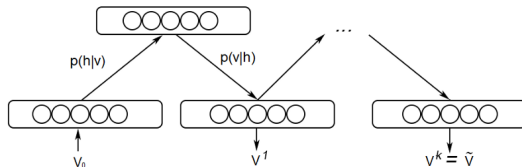
- ▶ Step 1, Gibbs sampling: what we have done in steps 1-6.
- ▶ Given an input vector  $\mathbf{v}$ , compute  $p(\mathbf{h}|\mathbf{v})$ .





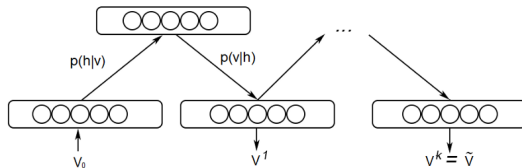
# RBM Training (1/2)

- ▶ **Step 1, Gibbs sampling:** what we have done in **steps 1-6**.
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# RBM Training (1/2)

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- ▶ Knowing the hidden values  **$\mathbf{h}$** , we use  **$p(\mathbf{v}|\mathbf{h})$**  for prediction of new input values  **$\mathbf{v}$** .
- ▶ This process is repeated  **$k$**  times.





## RBM Training (2/2)

- ▶ Step 2, **contrastive divergence**: what we have done in step 7.
  - Just a fancy name for **approximate gradient descent**.

$$\mathbf{w} = \mathbf{w} + \eta(\text{positive}(\mathbf{e}) - \text{negative}(\mathbf{e}))$$



# More Details about RBM



## Energy-based Model (1/3)

- ▶ **Energy** a quantitative property of **physics**.

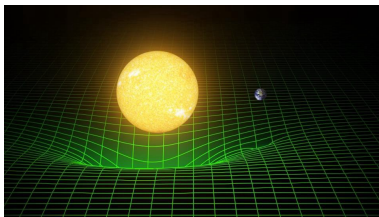
## Energy-based Model (1/3)

- ▶ **Energy** a quantitative property of **physics**.
  - E.g., **gravitational energy** describes the potential **energy** a **body with mass** has in relation to **another massive object** due to **gravity**.



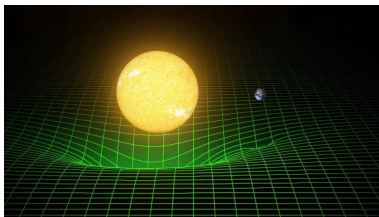
## Energy-based Model (2/3)

- ▶ One purpose of deep learning models is to **encode dependencies between variables**.



## Energy-based Model (2/3)

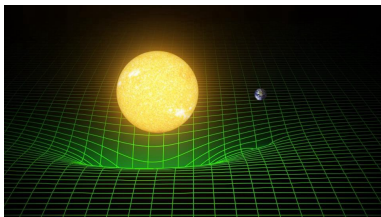
- ▶ One purpose of deep learning models is to **encode dependencies between variables**.
- ▶ The capturing of **dependencies** happen through associating of a **scalar energy** to each **state** of the **variables**.
  - Serves as a **measure of compatibility**.





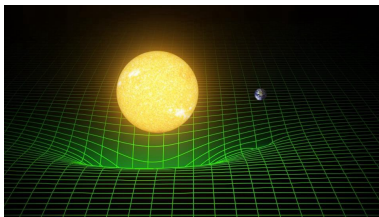
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- ▶ The capturing of **dependencies** happen through associating of a **scalar energy** to each **state** of the **variables**.
  - Serves as a **measure of compatibility**.
- ▶ A **high energy** means a **bad compatibility**.



## Energy-based Model (2/3)

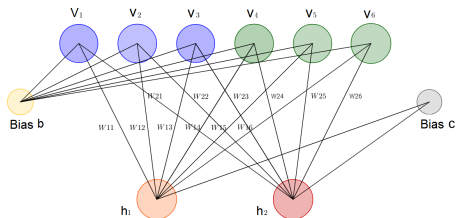
- ▶ One purpose of deep learning models is to **encode dependencies between variables**.
- ▶ The capturing of **dependencies** happen through associating of a **scalar energy** to each **state** of the **variables**.
  - Serves as a **measure of compatibility**.
- ▶ A **high energy** means a **bad compatibility**.
- ▶ An **energy based model** tries always to **minimize a predefined energy function**.



# Energy-based Model (3/3)

- ▶ The **energy function** for the RBMs is defined as:

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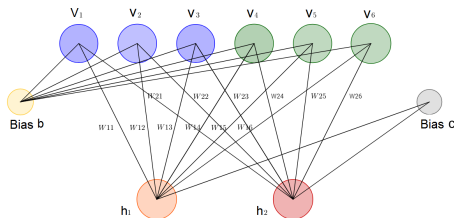


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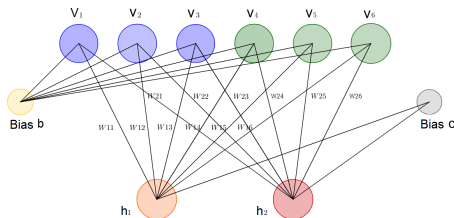


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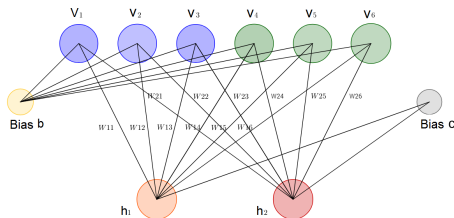


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- ▶ **b** and **c** are the **biases** of the visible and hidden layers, respectively.





## RBM is a Probabilistic Model (1/2)

- ▶ The probability of a certain state of  $\mathbf{v}$  and  $\mathbf{h}$ :

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- ▶ In physics, the joint distribution  $p(\mathbf{v}, \mathbf{h})$  is known as the Boltzmann Distribution or Gibbs Distribution.
- ▶ At each point in time the RBM is in a certain state.
  - The state refers to the values of neurons in the visible and hidden layers  $\mathbf{v}$  and  $\mathbf{h}$ .



## RBM is a Probabilistic Model (2/2)

- ▶ It is **difficult** to calculate the **joint probability** due to the **huge number of possible combination** of  **$\mathbf{v}$**  and  **$\mathbf{h}$** .

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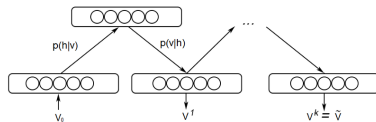
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- Much easier is the calculation of the **conditional probabilities** of state  **$\mathbf{h}$**  given the state  **$\mathbf{v}$**  and vice versa (**Gibbs sampling**)

$$p(\mathbf{h}|\mathbf{v}) = \prod_i p(h_i|\mathbf{v})$$

$$p(\mathbf{v}|\mathbf{h}) = \prod_j p(v_j|\mathbf{h})$$





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- ▶ Use the maximum-likelihood estimation.
- ▶ For a model of the form  $p(\mathbf{v})$  with parameters  $\mathbf{w}$ , the log-likelihood given a single training example  $\mathbf{v}$  is:

$$\log p(\mathbf{v}|\mathbf{h}) = \log \frac{\sum_{\mathbf{h}} e^{-E(\mathbf{v}, \mathbf{h})}}{\sum_{\mathbf{v}, \mathbf{h}} e^{-E(\mathbf{v}, \mathbf{h})}} = \log \sum_{\mathbf{h}} e^{-E(\mathbf{v}, \mathbf{h})} - \log \sum_{\mathbf{v}, \mathbf{h}} e^{-E(\mathbf{v}, \mathbf{h})}$$



## Learning in Boltzmann Machines (2/2)

- ▶ The log-likelihood gradients for an RBM with binary units:

$$\frac{\partial \log p(\mathbf{v}|\mathbf{h})}{\partial \mathbf{w}_{ij}} = \text{positive}(e_{ij}) - \text{negative}(e_{ij})$$





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- ▶ Then, we can update the weight  $\mathbf{w}$  as follows:

$$\mathbf{w}_{ij}^{(\text{next})} = \mathbf{w}_{ij} + \eta(\text{positive}(e_{ij}) - \text{negative}(e_{ij}))$$



# Summary



# Summary

- ▶ Autoencoders
  - Stacked autoencoders
  - Denoising autoencoders
  - Variational autoencoders
- ▶ Restricted Boltzmann Machine
  - Gibbs sampling
  - Contrastive divergence



## Reference

- ▶ Ian Goodfellow et al., Deep Learning (Ch. 14, 20)
- ▶ Aurélien Géron, Hands-On Machine Learning (Ch. 17)

Questions?